

# Negative Specific Heat of a Magnetically Self-Confined Plasma Torus

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## Abstract

It is shown that the thermodynamic maximum entropy principle predicts negative specific heat for a stationary magnetically self-confined current-carrying plasma torus. Implications for the magnetic self-confinement of fusion plasma are considered.

**CLASSIFICATION:**      PHYSICAL SCIENCES      Applied Mathematics

The goal of the controlled thermonuclear fusion program is to make the energy source that powers our Sun available to human society. Deep in the Sun's interior, favorable conditions for the quasi-stationary nuclear burning of the solar plasma prevail as a result of the immense gravitational self-forces that keep this huge accumulation of matter together. Since gravitational self-confinement is not operative at the reactor and laboratory scale, alternate means of confinement have to be employed to achieve sufficiently high plasma densities and temperatures in a reactor. In the perhaps most prominent stationary fusion reactor scheme, the tokamak, strong electric ring currents are induced in an electrically neutral plasma to achieve axisymmetric magnetic self-confinement in a rotationally invariant toroidal vessel  $\mathcal{T}$ . In a torus with sufficiently large major axis, such a magnetic self-confinement mimics gravitational self-confinement on account of the Biot-Savart law, according to which in a system of parallel current filaments all filaments attract each other magnetically with the same force law as would be the case gravitationally in a system of parallel mass filaments. What makes the magnetic forces more attractive (in the double sense of this phrase) than gravity for laboratory purposes is their very much bigger coupling constant ( $\approx 10^{40}v^2/c^2$  for two electrons moving on parallel trajectories with speed  $v$  as measured in the laboratory; note that the even stronger ( $v^2/c^2$  is replaced by 1) electrostatic repulsive forces between the same two electrons are very effectively screened in a neutral plasma and are traditionally neglected to a good approximation). Of course, the analogy does not extend to all aspects of plasma self-confinement. In particular, the solenoidal-vectorial character of stationary current densities necessitates the toroidal topology of magnetic self-confinement, whereas gravity not only allows but manifestly prefers spherical confinement over toroidal. Unfortunately, axisymmetric toroidal magnetic self-confinement is not known for its stability either. Although major efforts are devoted to the stabilization of the plasma configuration, a vast reservoir of instabilities capable of destroying the confinement has dramatically slowed down the development of a operating tokamak fusion reactor.

Matters are not exactly helped by the fact that our theoretical understanding of the physics on the various space-time scales that govern magnetic plasma confinement is still quite incomplete. In particular, while the solenoidal character of the magnetic induction together with the axisymmetry and stationarity of the law of momentum balance tell us that the poloidal magnetic flux function  $\Psi$  and the toroidal current density  $j$  must satisfy some local functional relation  $\mathcal{R}(\Psi, j, r) = 0$  (in which  $r$  is the cylindrical distance from the axis of symmetry) which turns Ampere's law into some in general nonlinear elliptic partial differential equation for  $\Psi$  (known in the fusion literature as the Grad-Shafranov equation), the actual relation  $\mathcal{R}$  is not fixed (except for the explicit  $r$ -dependence in  $\mathcal{R}$ ). Some information about  $\mathcal{R}$  should be contained in the law of energy balance between current drive (through an applied toroidal *emf* and other means), ohmic heating and, ultimately, radiation losses. Unfortunately only the so-called classical and neo-classical transport coefficients have been computed in some detail [1] whereas the small scale turbulent dissipation mechanisms in a tokamak plasma remain a largely challenging open problem. In this situation, theoreticians

have been forced to rely on fair judgment and good taste when guessing some additional principle(s) that would effectively complete the characterization of the stationary magnetically self-confined plasma torus in a tokamak.

A sizeable fraction of the literature employs a linear approximation of  $\mathcal{R}$  to get  $j \propto \Psi$ , rendering the Grad–Shafranov equation linear, and some improved-accuracy modeling uses a third-order polynomial approximation of  $\mathcal{R}$  [2]. Subsequently the filter of linearized dynamical stability analysis, based mostly on macroscopic magnetofluid theory and mesoscopic kinetic theory, is applied to sort out unstable configurations. While this approach has met with a certain limited success, one does not learn what the approximations are approximate to. Over the years a number of plasma theorists [3] have argued that an equilibrium thermodynamics-inspired maximum entropy principle with a few global dynamical constraints<sup>1</sup> should give answers close to the truth. In essence the various formulations in [3] give for  $\mathcal{R}$  the answer  $j \propto \exp(\Psi)$ , which leads to a nonlinear Grad–Shafranov equation that may have more than one solution  $\Psi$ , depending on the domain  $\mathcal{T}$ , the boundary conditions for  $\Psi$ , and the values of the physical parameters of the problem. In addition, this approach provides a global stability criterion within the class of axisymmetric states satisfying the same dynamical constraints. Only those solutions that maximize the relevant relative entropy functional will be globally stable.

Since in non-equilibrium statistical mechanics the maximum entropy principle has not acquired a status anywhere near as fundamental as in equilibrium statistical mechanics, it is mandatory to register some arguments in its favor for the case at hand. Thus, the relation  $j \propto \exp(\Psi)$  has been shown to be almost universally singled out also by a truly dissipative Fokker–Planck approach to stationary magnetically confined plasma [4, 5]. The perhaps most compelling reason to give it serious considerations, however, is the successful application of the maximum entropy approach to the physically distinct but mathematically quite similar problems of stationary planar incompressible flows,<sup>2</sup> where the vorticity plays a rôle closely analogous to the current density in the plasma torus, and the strongly magnetized pure electron plasma in a circular cylinder,<sup>3</sup> where the charge density plays that rôle. In this spirit, we have conducted a thorough investigation of the thermodynamic-type maximum entropy approach to the magnetically self-confined stationary plasma torus [5].

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<sup>1</sup>Since *any* probability density maximizes the entropy relative to itself, a stationary plasma torus is necessarily a maximum entropy configuration under *some* constraints. What makes the maximum entropy proposal non-empty is the insistence on only a few global natural dynamical constraints.

<sup>2</sup>The first qualitative predictions based on statistical mechanics of the Hamiltonian system of  $N$  point vortices were made in [6]. The quantitative evaluation began with [7]; impressive agreement with simulated flows is reported in [8]. Its mathematical rigorous foundations are by now almost complete, the latest word being [9]; see [10] for a review. More recently a formulation based directly on continuum vorticity has gained much ground; see [11] for a state-of-the-arts report.

<sup>3</sup>In the guiding center approximation the dynamics of this plasma system is identical to that of  $N$  point vortices [12]. Statistical mechanics in the corotating frame predicts that at high enough effective energies the nonlinear  $m = 1$  diocotron mode has higher entropy than any other configuration with the same energy and angular momentum [13], in accordance with remarkable real experiments [14].

A most curious finding of the study [5] is that the gravity-inspired toroidal magnetic plasma self-confinement scheme inherits from the stars their gravo-thermal *negative specific heat*.<sup>4</sup> This result is a little surprising, for it follows from what we said earlier that the plasma torus should actually more closely mimic a cylindrical caricature of a star, and the specific heat of a ‘cylindrical maximum entropy star’ [20] and its plasma physical clone, the cylindrical Bennett pinch [21], is non-negative! The existence of a maximum entropy plasma torus with negative specific heat is therefore a truly nontrivial fact. The purpose of this note is to point out some potentially important consequences of our finding for plasma physical applications.<sup>5</sup> To pave the way for the discussion we first describe the model and our results.

### The model

Since the whole problem is rotationally invariant, we work with conventional cylindrical coordinates  $r, \theta, z$ . The magnetic induction field decomposes accordingly as  $\mathbf{B} = \mathbf{B}_T + \mathbf{B}_P$ , where  $\mathbf{B}_T \parallel \mathbf{e}_\theta$  and  $\mathbf{B}_P \perp \mathbf{e}_\theta$ . In an actual tokamak, the toroidal component  $\mathbf{B}_T$  and a part,  $\mathbf{B}_0$ , of the poloidal component are externally generated harmonic fields that serve the purpose of azimuthal and radial stabilization. The total poloidal induction  $\mathbf{B}_P = \nabla\Psi \times \nabla\theta$  is the sum of  $\mathbf{B}_0$  and a component which is generated by the electric plasma current density vector,  $j\mathbf{e}_\theta$ , via the toroidal Ampère’s law

$$-r\nabla \cdot (r^{-2}\nabla\Psi) = 4\pi c^{-1}j. \quad (1)$$

In a similar manner one decomposes the electric field,  $\mathbf{E} = \mathbf{E}_T + \mathbf{E}_P$ , where  $\mathbf{E}_T$  is driving the plasma current while the poloidal part is determined by Coulomb’s law  $\nabla \cdot \mathbf{E} = 4\pi\rho$ , where  $\rho$  is the electric charge density. Usually the so-called quasi-neutrality approximation is invoked, which determines the poloidal electric field in leading order through a singular perturbative approach to Coulomb’s law. To keep matters as simple as possible, we consider an ‘electron-positron’ plasma, which is totally charge symmetric with regard to the particle

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<sup>4</sup>Ref. [15], pp. 60-63, explains why a homogeneous piece of “everyday matter” must have positive specific heat. See p. 62 of the same reference for why those arguments do not rule out negative specific heat in an isolated gravitating system. Indeed, the virial and the equipartition theorems imply that in a spherical equilibrium system the energy is distributed -2:1 between gravitational and kinetic. A decrease in total energy  $E$  of a gravitational equilibrium gas ball will increase its thermal energy. Such a system grows hotter while losing energy through, say radiation. Negative specific heat in self-gravitational perfect gases is evaluated quantitatively already in [16] and is further discussed in [17]; however, none of these configurations with negative specific heat is thermodynamically stable though some are metastable. Thermodynamically stable self-gravitating configurations with negative specific heat can occur when the Newtonian  $-1/r$  singularity is stabilized either as in quantum mechanics [18] or in classical hard balls systems [19].

<sup>5</sup>Recently, the existence and importance of negative specific heat was also reported for the diocotron mode of the guiding center plasma [13] alias point vortex gas, and for certain vorticity structures in geostrophic flows [11, 22]. However, very different from the gravo-thermal type negative specific heat that we report here to be a characteristic also of the magnetically self-confined plasma torus, the negative specific heat of these quasi-particle systems does *not* couple to the thermal motion of the underlying physical particle systems, which is evident from the fact that these quasi-particle systems also exhibit negative temperature [6].

species so that  $\rho$  vanishes exactly. In that case the poloidal electric field vanishes identically, too, while the toroidal one is implicitly contained in the electric plasma current  $I = Nq\omega/2\pi$ . Here  $N$  is total number of plasma particles,  $q$  the elementary charge, and  $\omega$  the mean absolute angular frequency of a species. This settles the electromagnetic part of the model, and we turn to the statistical mechanics part.

While most works in [3] are formulated at the macroscopic magnetofluid level, they can in essence be recovered from the statistical mechanics approach of Kiessling et al. in [3], which begins with the Hamiltonian  $N$  particle formulation and takes the kinetic limit. At this kinetic level, the plasma particles are described by distribution functions on single-particle phase space. We seek those distribution functions which maximize the familiar Boltzmann entropy functional under the constraints that the two separating integrals of motion, particle number and energy, take prescribed values  $N$  and  $E$ , and given that the plasma carries a prescribed electric current  $I$ . Since the current is not a separating integral, one has to resort to a ruse and prescribe each species' canonical angular momentum, which in the axisymmetric kinetic limit is a separating integral, and subsequently pass from this microcanonical angular momenta ensemble to its canonical convex dual, characterized by prescribed  $\omega$ , viz.  $I$ . The solutions of the corresponding Euler–Lagrange equations for this variational principle are also stationary solutions to the axisymmetric kinetic equations of Vlasov. Over velocity space the resulting distribution functions are simply rigidly rotating Maxwellians with temperature  $T = (k_B\beta)^{-1} > 0$ , tied to the energy constraint, and angular frequencies  $\pm\omega$  ( $\propto I$ ), microcanonically tied to the angular momentum constraints but canonically prescribed. This allows one to explicitly integrate over the velocity space to retain only the effective macroscopic entropy principle for the total number density of plasma particles,  $n(\mathbf{x}) = \bar{n}(r, z)$ , which in our charge symmetric plasma is just twice the value of the common space-dependent Boltzmann factor of each species' distribution function. In effect this renders the entropy a functional of  $n$ ,

$$S(n) = -k_B \int n(\mathbf{x}) \ln (\lambda_{\text{dB}}^3 n(\mathbf{x})/2) d\mathcal{T} + \frac{5}{2} N k_B, \quad (2)$$

where  $\lambda_{\text{dB}} = h/\sqrt{2\pi m k_B T}$  is the thermal de Broglie wave length. This entropy functional has to be maximized under the constraints that  $n \geq 0$  is axisymmetric,  $\int n d\mathcal{T} = N$ , and that the effective energy functional<sup>6</sup>

$$W(n) = -\frac{1}{2} \iint n(\mathbf{x}) K(\mathbf{x}, \mathbf{x}') n(\mathbf{x}') d\mathcal{T} d\mathcal{T}' + \frac{3}{2} N k_B T \quad (3)$$

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<sup>6</sup>The negative sign in front of the magnetic energy in  $W$  is due to the canonical constraint of prescribed electric current; cf. the negative sign in front of the centrifugal contribution to the kinetic energy of a rotating thermal system in the co-rotating frame, see Landau-Lifshitz (op. cit., pp. 71-73). Incidentally, those very centrifugal contributions to  $W$  are negligible in our plasma and have been omitted. Moreover, the toroidal field  $\mathbf{B}_T$  does not show since we consider only axisymmetric configurations with toroidal current density.

takes a prescribed value, say  $E$ . Here,  $K(\mathbf{x}, \mathbf{x}') = (2\pi I/cN)^2 G(\mathbf{x}, \mathbf{x}')$ , and  $G$  is the Green's function for  $-\nabla \cdot (r^{-2} \nabla)$  in  $\mathcal{T}$  for boundary conditions detailed below. With the help of a variant of Moser's corollary [23] of the Trudinger–Moser inequality it can be shown that given  $I$ , the entropy functional  $S(n)$  takes its finite maximum on the set of non-negative axisymmetric densities  $n(\mathbf{x}) = \bar{n}(r, z)$  satisfying  $\int n(\mathbf{x}) d\mathcal{T} = N > 0$  and  $W(n) = E$ , and the maximizer is a regular solution of the Euler–Lagrange equation.<sup>7</sup> We remark that more than one maximizing density function  $n$  might exist, and in addition the nonlinear Euler–Lagrange equation may have other types of solutions. We call a solution  $S$  *stable* if it is a *global* maximizer of the entropy (for the given constraints), *meta-stable* if it is merely a *local* maximizer, and unstable otherwise. Of course, only those  $S$  stable solutions which exhibit magnetic self-confinement are of interest.

## Results

Explicitly carrying out the variations and converting the Euler–Lagrange equation for  $n$  into an equation for  $\Psi$ , using  $\Psi(\mathbf{x}) = c^{-1} \int G(\mathbf{x}, \mathbf{x}') j(\mathbf{x}')/r' d\mathcal{T}'$  and  $j(\mathbf{x}) = qn(\mathbf{x})\omega r$ , we obtain Pfirsch's [24] nonlinear Grad–Shafranov equation

$$-\nabla \cdot (r^{-2} \nabla \Psi) = 8\pi^2 c^{-1} I \frac{e^{\beta\omega q\Psi/c}}{\int e^{\beta\omega q\Psi/c} d\mathcal{T}}, \quad (4)$$

which is to be solved in the torus  $\mathcal{T}$  for the boundary conditions encoded in  $G$ . Solving (4) is in general only possible numerically on a computer. However, some explicit analytical control is available if one simplifies the actual laboratory geometry somewhat and considers a torus  $\mathcal{T}$  with rectangular cross section  $\{r, \theta, z | r_i < r < r_o; \theta \text{ fixed}; 0 < z < H\}$ . The poloidal flux function  $\Psi(\mathbf{x}) = \bar{\Psi}(r, z)$  is assumed to satisfy periodic conditions at the  $z$  boundary and to be constant at  $r_i$  and  $r_o$ , so that the radial component of  $\mathbf{B}_P$  vanishes at the inner and outer boundaries of  $\mathcal{T}$ . In this setting the harmonic poloidal part is simply a homogeneous  $\mathbf{B}_0 \parallel \mathbf{e}_z$ , which we choose so that  $\bar{\Psi}(r_i, z) = \bar{\Psi}(r_o, z)$ . By gauge freedom we can now even set  $\bar{\Psi}(r_i, z) = 0$ . Beside the desired self-confined configurations, these boundary conditions allow also unconfined ones, namely Pfirsch's toroidal sheet pinch [24], given by the following  $\mathbf{e}_z$ -invariant solution of (4),

$$\Psi_{\text{Pf}}(r) = -\frac{2c}{\beta\omega q} \ln \frac{\cosh(\kappa^2[2r^2 - r_o^2 - r_i^2]/2)}{\cosh(\kappa^2[r_o^2 - r_i^2]/2)}, \quad (5)$$

with  $\kappa \in (0, \infty)$  a parameter and  $\beta(\kappa^2)$  given by

$$\beta = 4c^2 q^{-2} N^{-1} \omega^{-2} H \kappa^2 \tanh(\kappa^2[r_o^2 - r_i^2]/2). \quad (6)$$

While these solutions do not describe a magnetically self-confined plasma torus, they serve as our jumping off point for the numerical computations of the confined configurations. Our

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<sup>7</sup>These are quite nontrivial facts. In particular, all this is not true if we relax the condition of axisymmetry.

strategy, which was also contemplated by K. Schindler,<sup>8</sup> is to look for ringlike bifurcations from the toroidal sheet solution (5). At an infinite sequence of discrete values  $E_1 > E_2 > \dots$ , with  $E_k \searrow E_\infty = -\pi^2 I^2 (r_o^2 - r_i^2) / 2Hc^2$ , other solutions bifurcate off of the sheet pinch sequence, breaking its  $z$  invariance. The bifurcation points are determined by setting  $\Psi(\mathbf{x}) = \Psi_{\text{Pf}}(r) + \epsilon\psi(r, z) + O(\epsilon^2)$ , with  $\psi(r_{i,a}, z) = 0$  and  $\psi(r, z + H) = \psi(r, z)$ , and expanding (4) to first order in  $\epsilon$ , giving the linearized problem

$$-\nabla \cdot (r^{-2} \nabla \psi) + V\psi - V\langle\psi\rangle = 0, \quad (7)$$

where  $\langle\psi\rangle = \int \psi(r, z) V(r) d\mathcal{T} / \int V(r) d\mathcal{T}$ , and

$$V(r) = -8\kappa^4 \text{sech}^2(\kappa^2[2r^2 - r_o^2 - r_i^2]/2). \quad (8)$$

By Fredholm's alternative [25], the solution of (7) is trivial except for certain discrete values of  $\kappa$  at which the bifurcations occur. We have proved [5] that for our  $\mathcal{T}$  all bifurcations off of the sheet pinch are due to modes  $\psi_k$ ,  $k = 1, 2, \dots$ , that satisfy  $\langle\psi_k\rangle = 0$ . The first mode is of the form  $\psi_1(r, z) = R(r) \cos(2\pi[z - z_0]/H)$ , with  $z_0$  arbitrary, and with  $R(r)$  satisfying

$$-r (r^{-1} R')' + (2\pi/H)^2 R + r^2 V R = 0 \quad (9)$$

for  $R(r_i) = R(r_o) = 0$ . With realistic domain dimensions  $r_i = 1$ ,  $r_o = \sqrt{2}$ , and  $H = 2$ , a standard Runge-Kutta solver finds the unique nontrivial solution at  $\kappa = \kappa_1 = 1.62$ , giving  $E_1 = 2.72 W_\bullet$ , with  $W_\bullet = 2\pi^2 r_i I^2 / 25c^2$ . Numerical solutions of (4), with  $r_i = 1$ ,  $r_o = \sqrt{2}$ , and  $H = 2$ , were then computed with a well-tested bifurcation code [26], based on a continuation method [27]. Our code reproduced the analytical sheet pinch solution and its first bifurcation point in excellent agreement with our independently obtained semi-analytical results. We then numerically followed the first bifurcating branch that emerges from  $\psi_1$  to nonlinear amplitudes, where it develops into a toroidal ring pinch with a double X magnetic structure similar to the double X structure in the PDX-PBX Tokamak experiment in Princeton.<sup>9</sup> Retrospectively, this vindicates our choice of boundary conditions.

(FIG. 1)

Our primary interest is in the energy-entropy diagram. Shown in Fig. 2 is  $\Delta S(n)$  versus  $W(n)$ , where  $\Delta S(n) = S(n) - Nk_B \ln \sqrt{(4\pi e/N)^5 (I/\hbar c)^6 m^3 r_i^3}$ , with  $e$  the Euler number, and with the density function  $n$  running along the computed bifurcation sequences of ring and sheet pinch. At sufficiently high effective energies, Pfirsch's sheet pinch is the unique solution of (4) for the stipulated boundary conditions, hence maximizing entropy. Numerically it appears to be the case for all  $W(n) > E_1 = 2.72 W_\bullet$ , see Fig. 2. For all  $W(n) < E_1$  down to  $W(n) = -0.5 W_\bullet$  where we terminated the computation, the ring pinch has higher entropy than the toroidal sheet pinch at the same effective energy. By asymptotic analysis we found that also for  $W(n) \searrow -\infty$ , and by continuity for  $W(n) \ll -W_\bullet$ , the maximum entropy

<sup>8</sup>Private communication.

<sup>9</sup>For design information, see <http://www.pppl.gov/oview/pages/pbxm-design.html>.

configuration consists of a highly concentrated ring pinch which, in rescaled coordinates centered at the density maximum, converges to Bennett's cylindrical pinch [21] as  $W(n) \searrow -\infty$ . On the basis of this evidence we surmise that the ring pinch has maximum entropy for all  $W(n) < E_1$ , implying its  $S$  stability in the class of rotationally invariant plasma with effective energy  $W(n) < E_1$  and current  $I$ . We remark that the first bifurcation off of the toroidal sheet pinch into the  $S$  stable toroidal ring pinch branch is then a symmetry-breaking second-order phase transition.

It remains to determine the specific heat of the configurations, which we recall is negatively proportional to the second derivative of  $S$  with respect to  $E$ . Thus we inspect the curvature of the graphs of the entropy as function of energy for the various solutions, given in Fig. 2. The graph representing the sheet pinch is concave. However, the graph for the ring pinch is manifestly convex over the whole computed range of energies  $-0.5 W_\bullet < W(n) < E_1$ . We were also able to prove the convexity analytically to second order in perturbation theory away from the bifurcation point. This confirms what we have announced earlier: *the specific heat of the ring pinch is negative!*

(FIG. 2)

## Discussion

At last, we discuss the potential implications that our finding of gravo-thermal type negative specific heat has for the problem of toroidal magnetic self-confinement of plasma. In a gravitationally bound plasma, negative specific heat on one hand aids the ignition of nuclear burning in a proto-star by heating it up when it loses energy by radiation, but it is also responsible for some more spectacular instabilities once the nuclear burning expires, like the onset of the red giant structure [17]. It would be intriguing enough if the negative specific heat of a magnetically self-confined plasma torus should be confirmed to aid the ignition of nuclear burning in a tokamak. For this to be so, one would have to be able to hold  $N$  and  $I$  fixed and secure the toroidal invariance (which is what one wants to achieve anyhow), while  $E$  would have to decrease (the plasma radiation would seem to help in this respect) *slow enough* so that one would essentially evolve along the ring pinch branch in Fig.2 to the left, thereby heating up the plasma while pinching it more strongly. This would not seem unwelcome. For now, however, energy leakage by radiation is a serious problem, while at the same time the *emf* current drive leads to yet uncontrolled ohmic heating of the plasma. In this case where  $E$  is allowed to fluctuate too widely, the negative specific heat will have a very unwanted effect on the confinement. This can be illustrated by considering the temperature rather than energy  $E$  to be controlled by the competition of ohmic heating and radiation (still assuming  $N$  and  $I$  fixed, and toroidal invariance). In that case the canonical ensemble determines the stability. But microcanonical and canonical ensembles are not equivalent when the microcanonical one exhibits states with negative specific heat [18, 9, 28, 22], and sure enough, none of the computed ring pinches with negative specific heat minimizes the



free energy functional

$$F(\Psi) = -\frac{1}{8\pi} \int r^{-2} |\nabla \Psi|^2 d\mathcal{T} + N\beta^{-1} \ln \left( \frac{2e}{N\lambda_{\text{dB}}^3} \int e^{\beta\omega q\Psi/c} d\mathcal{T} \right). \quad (10)$$

Actually,  $F$  is unbounded below for these  $\beta$ ,  $N$  and  $I$  values, any minimizing sequence concentrating on a singular ring current; cf. [29] for a good discussion of the translation-invariant analog. Of course, a real plasma would not get anywhere near such a singular ring current configuration, for a highly concentrated plasma ring is known to be susceptible to magnetofluid dynamical instabilities that destroy the axisymmetry.

### Conclusion

To summarize, the  $S$  stable ring pinches have negative specific heat of the gravo-thermal type and will therefore be stable *if and only if*  $N$ ,  $E$ , and  $I$  are essentially fixed and the toroidal invariance is secured, in which case the negative specific heat may aid the ignition of thermonuclear burning. The “if” part is good news; the bad news is the “only if” part.

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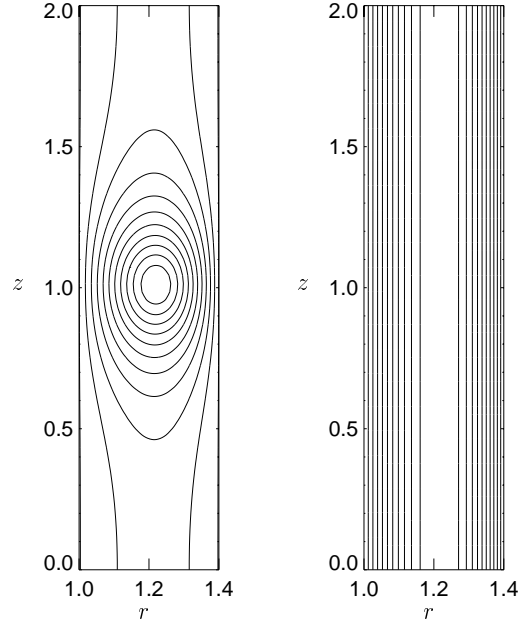


Figure 1: Poloidal magnetic lines of force of maximum entropy solutions near the second order phase transition at  $E_1 = 2.72 W_\bullet$ . Ring pinch (left):  $W(n) = 2.34 W_\bullet$ ,  $\beta = 0.29N/W_\bullet$ ,  $z_0 = 0$ ; Sheet pinch (right):  $W(n) = 3.00 W_\bullet$ ,  $\beta = 0.30N/W_\bullet$ . The toroidal hoop effect is neatly visible.

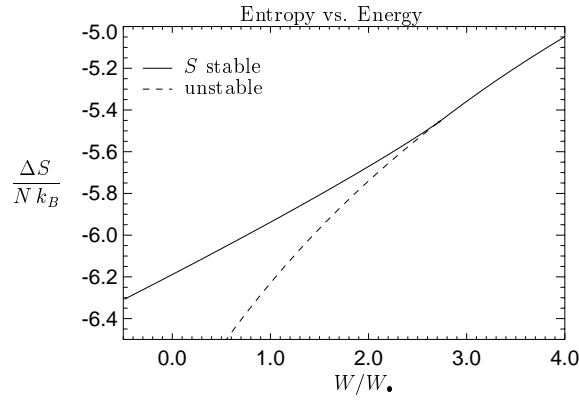


Figure 2:  $\Delta S(n)$  versus  $W(n)$  for toroidal sheet pinch (concave branch) and ring pinch (convex branch).